Gravitation with superposed Gauss–Bonnet terms in higher dimensions: Black hole metrics and maximal extensions[*]

A. Chakrabarti[†]

Centre de Physique Théorique[‡], Ecole Polytechnique, F-91128 Palaiseau, France

D. H. Tchrakian[§]

Department of Mathematical Physics, National University of Ireland Maynooth, Maynooth, Ireland
and
School of Theoretical Physics – DIAS, 10 Burlington Road, Dublin 4, Ireland

Our starting point is an iterative construction suited to combinatorics in arbitarary dimensions d, of totally anisymmetrised p-Riemann $2p$-forms ($2p \leq d$) generalising the (1-)Riemann curvature 2-forms. Superposition of p-Ricci scalars obtained from the p-Riemann forms defines the maximally Gauss–Bonnet extended gravitational Lagrangian. Metrics, spherically symmetric in the $(d-1)$ space dimensions are constructed for the general case. The problem is directly reduced to solving polynomial equations. For some black hole type metrics the horizons are obtained by solving polynomial equations. Corresponding Kruskal type maximal extensions are obtained explicitly in complete generality, as is also the periodicity of time for Euclidean signature. We show how to include a cosmological constant and a point charge. Possible further developments and applications are indicated.

arXiv:hep-th/0101160v2 18 Apr 2001

I. INTRODUCTION

Higher order terms in the Riemann tensor appear in the gravitational sector of string theory [1]. Here, we choose to consider only the Gauss-Bonnet (GB) terms of all orders which assure that the Lagrangian contains only quadratic powers of the velocity fields (i.e. the derivatives of the metric or the vielbein). For the second order (in the Riemann curvature) terms, which will be labeled by $p = 2$ in the following, it was indeed shown [29] that the corresponding terms arising in the string theoretic context can be reduced to the GB form by suitably redefining the fields. GB extended Einstein equations in higher dimensions have been studied by various authors since a long time in various contexts [3–6], namely that of cosmological solutions [5], gravitational instantons [3,4] and black holes [5,6]. Of these, the work of [6] is the closest to our present work in that all possible GB terms are taken into account.

Recently we have studied black hole solutions of generalised gravitational systems consisting of single Gauss–Bonnet terms, considered as members of a hierarchy of generalised gravitational systems, each labeled with an integer p corresponding to the $2p$-form Riemann curvature defining it. Each member is the p-Ricci scalar $R_{(p)}$ formed by the contraction of the indices on the $2p$-form Riemann tensor. These described black hole vacuum metrics [7], and metrics with point charge [8] generalising the Reissner–Nordstrom solutions. In the present work we present a particularly convenient and systematic formalism for a Lagrangian consisting of the superposition of these individual p-Ricci scalars, $R_{(p)}$ with constant dimensional coefficients $\kappa_{(p)}$,

$$\mathcal{L} \ = \ \sum_{p=1}^{P} \ \frac{1}{2p} \ \kappa_{(p)} \ R_{(p)} \ , \tag{1.1}$$

The systems considered must be in dimensions $d \ \geq \ 5$, and due to the antisymmetry of the $2p$-forms consist of P terms such that $2P \ \leq \ d$.

For spherical symmetry in $d - 1$ *space* dimensions with metric

$$ds^2 = \mp N(r)dt^2 \ + \ N(r)^{-1}dr^2 \ + \ r^2 d\Omega^2_{(d-2)} \ , \tag{1.2}$$

it is shown in Sections **III** and **IV**, that the metric pertaining to the system (1.1), generalising the standard Schwarzschild, Reissner–Nordstrom and deSitter soltions, is obtained by solving the polynomial equation for $(1 - N(r))$

$$\sum_{p=1}^{P} \ \frac{\kappa_{(p)}}{2^p p!} (d-2)(d-3)...(d-2p) \left(\frac{1-N}{r^2} \right)^p \ = \ \frac{c}{r^{d-1}} \ - \ \frac{b}{r^{2(d-2)}} + \lambda \tag{1.3}$$

This is our crucial result. For $P \ > \ 2$, elliptic and hyperelliptic functions (theta functions of suitably higher genus) are needed to construct the solutions explicitly. Relating the parameters characterising these functions to those appearing in (1.3) is in general a difficult task. However in principle a complete set of solutions can be obtained in each case, the required prescriptions [9] for which are available. Hence for spherical symmetry, the problem of the construction of the metric pertaining to the system (1.1) can be considered solved.

We will study the properties of some relatively simple cases in the following Sections. We will assume that in general in (1.3)

$$\kappa_{(1)} \ \gg \ \kappa_{(p)} \ , \qquad p \ > \ 1 \ .$$

Hence, having obtained the solution of (1.3) for the usual Einstein–Hilbert case, with $\kappa_{(p)} = 0$ for $p \ > \ 1$, one can consider systems consisting of a series of terms with coefficients $\kappa_{(2)}, \kappa_{(3)}, ...,$ with $\kappa_{(2)} \gg \kappa_{(3)}, ...$ In this sense, such systems could be considered as perturbative series, and successive terms are expected to become appreciable with increasing energies. For a fixed number of $\kappa_{(p)}$ our results *per se* are, in principle, exact. Let us consider an example of interest, namely the horizons, $r = r_H$. These are by definition obtained setting $N(r_H) = 0$ in (1.3), and then solving for r_H. In

constructing exact solutions of polynomial equations for horizons in Section **V**, out of the full set of solutions one can select the real positive one by comparing with perturbative (in $\kappa_{(p)}$, $p > 1$) solutions. Consider for simplicity the Schwarzschild–like case, with $b = 0 = \lambda$, with moreover $\kappa_{(p)} = 0$ for $p > 2$. One can first, setting $\kappa_{(2)} = 0$, obtain the real positive horizon $r_H^{(0)}$, and then for small nonzero $\kappa_{(2)}$ obtain

$$r_H = r_H^{(0)} + o(\kappa_{(2)}) + o(\kappa_{(2)}^2) \ .$$

Consistency with this perturabtive series solution will select out the real positive horizon r_H from the exact solutions of the relevant polynomials (1.3). Examples will be given in Section **V**.

In contrast to our considerations in Section **V**, where we truncated the values of p to lower than the maximum possible value P (consistent with $2P < d$), we emphasise that our formalism yields some exact results (for systems featuring all $\kappa_{(p)}$ up to $\kappa_{(P)}$). One such case concerns maximal extensions in Section **VI**. We assume $r = r_H$ to be an exact solution of (1.3) with $N(r_H) = 0$ and then for

$$r = r_H + \rho \ , \qquad \rho \ll 1 \ ,$$

we set

$$N(\rho) = 2\delta\rho + o(\rho^2)$$

We obtain an exact solution for δ, *viz.* (6.3), as a function of r_H, the *space-time* dimension d, and the $\kappa_{(p)}$. The parameter δ plays a crucial role concerning the near–horizon geometry, the periodicity of the Euclidean time, and the Hawking temperature. The exact general solution (6.3) is hence of considerable iterest.

In Section **VII** we have indicated how our results of Section **V** can be extended to include metrics pertaining to systems with Maxwell and cosmological terms added to (1.1), i.e. the construction of Reissner–Nordstrom and deSitter metrics.

Possibilities of generaisations and applications of our present study are discussed in our conclusions in section **VIII**.

II. GENERALISED LAGRANGIAN AND EINSTEIN TENSOR

Let e^a be the tangent frame vector 1-forms

$$e^a = e_\mu^a \, \mathrm{d}x^\mu \ , \tag{2.1}$$

where as usual $a, b, ...$ denote frame indices and $\mu, \nu, ...$ space-time ones, and ω^{ab} the antisymmetric Levi-Civita spin-connection 1-forms

$$\omega^{ab} = \omega_\mu^{ab} \mathrm{d}x^\mu = -\omega^{ba} \ , \tag{2.2}$$

satisfying

$$\mathrm{d}e^a + \omega^{ab} \wedge e_b = 0. \tag{2.3}$$

The curvature 2-forms are then

$$\begin{aligned} R^{ab} &= \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = -R^{ba} \ , \\ &= R_{\mu\nu}^{ab} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = R_{a'b'}^{ab} \, e^{a'} \wedge e^{b'}. \end{aligned} \tag{2.4}$$

We will often use the last form involving only tangent plane indices.

Starting with R^{ab} higher order p-form terms, totally antisymmetrised in the indices $a, b, c, ...$, are defined iteratively as follows

3

$$R^{abcd} = R^{ab} \wedge R^{cd} + R^{ad} \wedge R^{bc} + R^{ac} \wedge R^{db} , \qquad (2.5)$$

$$R^{a_1 a_2 ... a_{2p}} = R^{a_1 a_2} \wedge R^{a_3 a_4 ... a_{2p}} + \text{cyclic permutations of } (a_2, a_3, ..., a_{2p}) , \qquad (2.6)$$

for the $p = 2$ and the generic p cases, respectively. The antisymmetric $2p$-form curvature (2.6) consists of $3.5...(2p-3)(2p-1)$ terms of the type

$$R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge ... \wedge R^{a_{2p-1} a_{2p}} .$$

For $p = 1$ (2.6) coincides with (2.4), the usual curvature, in any dimension d. For $d < 2p$, (2.6) vanishes due to the antisymmetry. For $d = 2p$, (2.6) becomes the topological Euler density in those dimensions. For odd dimensions, $p = \frac{1}{2}(d-1)$ in (2.6) leads to features analogous to that in $d = 3$ for (2.4).

One can express (2.6) in *vielbein* basis, generalising (2.4), as

$$R^{a_1 a_2 ... a_{2p}} = R^{a_1 a_2 ... a_{2p}}_{b_1 b_2 ... b_{2p}} e^{b_1} \wedge e^{b_2} ... \wedge e^{b_{2p}}. \qquad (2.7)$$

The p-Ricci tensor is then defines as

$$R_{(p)}{}^{a_1}_{b_1} = \sum_{(a_2, ..., a_{2p})} R^{a_1 a_2 ... a_{2p}}_{b_1 a_2 ... a_{2p}} , \qquad (2.8)$$

and the p-Ricci scalar as

$$R_{(p)} = \sum_a R_{(p)}{}^a_a . \qquad (2.9)$$

The generalised Einstein–Hilbert Lagrangian is now defined to be

$$\mathcal{L} = \sum_{p=1}^{P} \frac{1}{2p} \kappa_{(p)} R_{(p)} , \qquad (2.10)$$

where $2P = d$ and $d - 1$, respectively, for even and odd d. Each coupling constant $\kappa_{(p)}$ is taken to be positive and has dimension $(\text{length})^{2p}$, rendering the Lagrangian dimensionless. For

$$\kappa_{(2)} = \kappa_{(3)} = ... = \kappa_{(P)} = 0$$

one recovers the usual Einstein–Hilbert Lagrangian in dimension d. We will always set

$$\kappa_{(1)} > 0 .$$

Some or all of the others, $\kappa_{(p)}$, $p = 2, 3, .., P$, can then be chosen to be non-zero. We will choose at least $\kappa_{(2)} > 0$ so as to illustrate higher–order effects, and $\kappa_{(1)}$ will be taken to be much larger than $\kappa_{(p)}$, $p = 2, 3, .., P$, so that the latter terms can be considered to play a perturbative role.

For each p, the p-Einstein tensor is defined as

$$G_{(p)}{}^a_b = R_{(p)}{}^a_b - \frac{1}{2p} \eta^a_b R_{(p)} , \qquad (2.11)$$

and for the system (2.10) it is

$$G^a_b = \sum_{p=1}^{P} \frac{1}{2p} \kappa_{(p)} G_{(p)}{}^a_b . \qquad (2.12)$$

III. SPHERICAL SYMMETRY

We impose spherical symmetry in the $d-1$ space–dimensions by requiring the diagonal metric, for Lorentz and Euclidean signatures, respectively

$$ds^2 = \mp N(r)dt^2 \; + \; N(r)^{-1}dr^2 \; + \; r^2 d\Omega^2_{(d-2)} \;, \tag{3.1}$$

where

$$d\Omega^2_{(d-2)} = \; d\theta_1^2 \; + \; \sin^2\theta_1 \; d\theta_2^2 \; + ... + \left(\prod_{n=1}^{d-3}\sin\theta_n\right)^2 d\theta_{d-2}^2 \;.$$

We shall henceforth use the following notation[1] for the frame indices

$$x^a \;=\; (t, r, \theta_1, \theta_2, ..., \theta_{d-2}) \;=\; (t, r, 1, 2, ..., d-2) \;,$$

so that for diagonal metrics one can set (with no summation)

$$e^a \;=\; \sqrt{|g_{aa}|} \; \mathrm{d}x^a \;. \tag{3.2}$$

The consequent simplifying properties [7] of the spin–connections lead finally to, using labeling with $i, j = 1, 2, ..., d-2$, and the notation

$$N(r) \;=\; 1 \;-\; L(r) \;, \quad \text{with} \quad L' = \frac{dL}{dr} \;, \tag{3.3}$$

$$R^{tr} = \frac{1}{2} \; L'' \; e^t \wedge e^r \;, \qquad R^{ti} \;=\; \frac{1}{2r} \; L' \; e^t \wedge e^i \;,$$

$$R^{ri} = \frac{1}{2r} \; L' \; e^r \wedge e^i \;, \qquad R^{ij} \;=\; \frac{1}{r^2} \; L \; e^i \wedge e^j \;. \tag{3.4}$$

IV. METRICS

From the results of the previous two Sections, namely by applying (3.4) to (2.11), we arrive at the remarkably compact expressions for the non-vanishing components of G^a_b below

$$G^t_t = G^r_r \;=\; -\left(r\frac{d}{dr} + (d-1)\right)V(r) \tag{4.1}$$

$$G^i_i = -\frac{1}{d-2}\left(r\frac{d}{dr} + (d-2)\right)\left(r\frac{d}{dr} + (d-1)\right)V(r) \;, \tag{4.2}$$

with $i = 1, 2, .., d-2$, and where

$$V(r) \;=\; \sum_{p=1}^{P} \kappa_{(p)} \frac{(d-2)(d-3)...(d-2p)}{2^p p!}\left(\frac{L}{r^2}\right)^p \;. \tag{4.3}$$

---

[1]We do not distinguish the tangent plane indices from the from the coordinate ones $(t, r, 1, 2, ..., d-2)$, for example by introducing yet another notation $(\hat{t}, \hat{r}, \hat{1}, \hat{2}, ...)$. For diagonal metrics, this simplifying notation does not cause ambiguities. As tangent plane indices they are raised and lowered using $\eta^a{}_b$ rather than $g^\mu{}_\nu$.

5

It is clear that the term with $d = 2p$ vanishes, so that the summation in (4.3) runs up to $2P < d$.

All p dependence is contained in $V(r)$ which is a polynomial in $\frac{L}{r^2}$. Hence the constraints on $V(r)$ itself, namely the variational equations, are independent of p and are the same as for the $p = 1$ case. Once these dynamical equations are solved, the next step is to solve a polynomial equation in $\frac{L}{r^2}$. This result is crucial.

Consistently with spherical symmetry one can set

$$V(r) \ = \ \frac{c}{r^{d-1}} - \frac{b}{r^{2(d-2)}} + \lambda \tag{4.4}$$

when

$$G_t^t = G_r^r \ = \ -\frac{(d-3)b}{r^{2(d-2)}} - (d-1)\lambda \tag{4.5}$$

$$G_i^i = \ \frac{(d-3)b}{r^{2(d-2)}} - (d-1)\lambda \ . \tag{4.6}$$

The first term in (4.4), namely $cr^{-(d-1)}$, represents a vacuum solution leading to the generalised Schwarzschild type black hole [7]. This is annihilated by the operator $\left(r\frac{d}{dr} + (d-1)\right)$ present in each G_a^a in (4.1)-(4.2).

Adding the second term in (4.4), $-br^{-2(d-2)}$, leads to generalised Reissner–Nordstrom type [8] solutions in the presence of a point charge in d dimensions. Finally adding λ in (4.4) can include the presence of a cosmological constant.

Note the effect of the extra factor

$$\frac{1}{d-2}\left(r\frac{d}{dr} + (d-2)\right) \tag{4.7}$$

in (4.2) as compared to its absence in (4.1). In (4.6) this induces just a change of sign as compared to (4.5). This compensates precisely for the corresponding sign of the angular components T_i^i of the stress-energy tensor of a point charge q in d dimensions, namely

$$T_t^t \ = \ T_r^r \ = \ -T_i^i \ = -\frac{(d-3)q^2}{2r^{2(d-2)}} \ . \tag{4.8}$$

Acting on the third term λ, of (4.4), the action of the operator (4.7) yields *unity*. The constants b and λ are to be fixed, finally, after choosing suitable units, by inserting (4.5) and (4.6) in

$$G^a_{\ b} \ = \ const. \ T^a_{\ b} \ + \ \Lambda \ \delta^a_{\ b} \ . \tag{4.9}$$

In the following Sections we will give examples of explicit solutions for $L(r)$ using (4.3) and (4.4). We will start with Schwarzschild type black holes with $b = \lambda = 0$ and will study their properties.

V. GENERALISED SCHWARZSCHILD–TYPE BLACK HOLES

In the case of vanishing stress–energy tensor, (4.5) and (4.6) are solved by $V(r) = cr^{-(d-1)}$, with $b = 0 = \lambda$, yielding

$$\frac{\kappa_{(1)}}{2}(d-2)\left(\frac{L}{r^2}\right) + \frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)\left(\frac{L}{r^2}\right)^2 + ...$$

$$... + \frac{\kappa_{(P)}}{2^P P!}(d-2)(d-3)...(d-2P)\left(\frac{L}{r^2}\right)^P = \frac{c}{r^{d-1}} \tag{5.1}$$

with $2P < d$. With a single nonvanishing p, the solution of (5.1) reduces to the function $L(r)$ desribing the p-Schwarzschild metric of [7], and with $p = 1$ to the d-dimensional Schwarzschild metric of [10].

For a horizon, denoted by $r = r_H$, by definition

$$N(r_H) = 1 - L(r_H) = 0 \quad \mapsto \quad L(r_H) = 1 . \tag{5.2}$$

Hence

$$\frac{\kappa_{(1)}}{2}(d-2)\,(r_H)^{-2} + \frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)\,(r_H)^{-4} + \ldots$$
$$\ldots + \frac{\kappa_{(P)}}{2^P P!}(d-2)(d-3)\ldots(d-2P)\,(r_H)^{-2P} = c\,(r_H)^{-(d-1)} . \tag{5.3}$$

We will look for *positive real roots* only.

Let us now look at particular cases to better understand the possibilities. In practice we will restrict to the case $\kappa_{(3)} = \kappa_{(4)} = \ldots = \kappa_{(P)} = 0$, keeping only $\kappa_{(1)}$ and $\kappa_{(2)}$. One obtains, for $d > 4$,

$$L(r) = \left(\frac{2\kappa_{(1)}}{(d-3)(d-4)\kappa_{(2)}}\right)\left[\left(1 + \frac{2c(d-3)(d-4)\kappa_{(2)}}{(d-2)\kappa_{(1)}^2 r^{(d-1)}}\right)^{\frac{1}{2}} - 1\right] r^2 . \tag{5.4}$$

For $\kappa_{(2)} > 0$ this is real and positive. For $\kappa_{(2)} = 0$ it reduces to the usual Schwarzschild solution in d dimensions [10]. In all cases one obtains asymptotically flat solutions.

For the horizon, one has

$$\frac{\kappa_{(1)}}{2}(d-2)\,(r_H)^{-2} + \frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)\,(r_H)^{-4} = c\,(r_H)^{-(d-1)} . \tag{5.5}$$

(i) For dimension $d = 5$

$$r_H^2 = \frac{2}{3\kappa_{(1)}}\left(c - \frac{3}{4}\kappa_{(2)}\right) \tag{5.6}$$

Hence for $c > \frac{3}{4}\kappa_{(2)}$, there is a single real horizon at

$$r_H = \left(\frac{2}{3\kappa_{(1)}}\right)^{\frac{1}{2}}\left(c - \frac{3}{4}\kappa_{(2)}\right)^{\frac{1}{2}} \tag{5.7}$$

Compare this with the case $p = 2$ in [7] when $\kappa_{(1)} = 0$.

(ii) For dimension $d = 6$

$$r_H^3 + \frac{3}{2}\frac{\kappa_{(2)}}{\kappa_{(1)}}r_H - \frac{c}{2\kappa_{(1)}} = 0 . \tag{5.8}$$

Hence

$$r_H = (\alpha + \beta) , \quad \left(\alpha e^{i\frac{2\pi}{3}} + \beta e^{-i\frac{2\pi}{3}}\right) , \quad \left(\alpha e^{-i\frac{2\pi}{3}} + \beta e^{i\frac{2\pi}{3}}\right) \tag{5.9}$$

with

$$\alpha = \left(\frac{c}{4\kappa_{(1)}} + \left(\left(\frac{c}{4\kappa_{(1)}}\right)^2 + \left(\frac{\kappa_{(2)}}{2\kappa_{(1)}}\right)^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} ,$$

$$\beta = \left(\frac{c}{4\kappa_{(1)}} - \left(\left(\frac{c}{4\kappa_{(1)}}\right)^2 + \left(\frac{\kappa_{(2)}}{2\kappa_{(1)}}\right)^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} ,$$

7

which are real cube roots. Thus, for sufficiently small $\kappa_{(2)} \ll \kappa_{(1)}$, there is only one real horizon at

$$r_H = (\alpha + \beta) , \tag{5.10}$$

consistently with

$$r_H = \left(\frac{c}{2\kappa_{(1)}}\right)^{\frac{1}{3}} - \frac{1}{2}\frac{\kappa_{(2)}}{\kappa_{(1)}}\left(\frac{c}{2\kappa_{(1)}}\right)^{-\frac{1}{3}} + o\left(\kappa_{(2)}^2\right) .$$

(iii) For dimension $d = 7$

$$r_H^4 + 3\frac{\kappa_{(2)}}{\kappa_{(1)}} r_H^2 - \frac{2c}{5\kappa_{(1)}} = 0 , \tag{5.11}$$

giving the real positive value for the horizon

$$r_H = \left(\frac{3\kappa_{(2)}}{2\kappa_{(1)}}\right)^{\frac{1}{2}}\left[\left(1 + \frac{8c}{45}\frac{\kappa_{(1)}}{\kappa_{(2)}^2}\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}} . \tag{5.12}$$

(iv) For dimension $d = 8$

$$r_H^5 + 5\frac{\kappa_{(2)}}{\kappa_{(1)}} r_H^3 - \frac{c}{3\kappa_{(1)}} = 0 . \tag{5.13}$$

This is a quintic equation whose solution [9] can be expressed in terms of *elliptic functions*. Setting

$$r_H = \frac{a}{z}$$

(5.13) transforms into

$$z^5 - \left(\frac{5\kappa_{(2)}}{a^2\kappa_{(1)}}\right)\left(\frac{3a^5\kappa_{(1)}}{c}\right)z^2 - \left(\frac{3a^5\kappa_{(1)}}{c}\right) = 0 . \tag{5.14}$$

(5.14) is already a *principal quintic* [9] with in addition the linear term absent (vanishing coefficient of z). Moreover by choosing a suitably one can obtain a conveniently simple value for one of the two coefficients in (5.14) or for their ratio. These features simplify the task of explicitly constructing the solutions [9].

However since considerable more work is needed to realise these explicit solutions, we will not pursue them further here. We just add that the determination of the real positive root must be consistent with

$$r_H = \left(\frac{c}{3\kappa_{(1)}}\right)^{\frac{1}{5}} - \frac{\kappa_{(2)}}{\kappa_{(1)}}\left(\frac{c}{3\kappa_{(1)}}\right)^{-\frac{1}{5}} + o\left(\kappa_{(2)}^2\right) . \tag{5.15}$$

(v) For dimension $d = 11$

In this case there is a special simplification, namely that (5.5) reduces to a *quartic* in r_H^2, which permits an elementary solution. Thus (5.5) here is

$$r_H^8 + \frac{14\kappa_{(2)}}{\kappa_{(1)}} r_H^6 - \frac{2c}{9\kappa_{(1)}} = 0 . \tag{5.16}$$

This can first be solved as a quartic in r_H^2 and then the positive square root taken consistently with

$$r_H \;=\; \left(\frac{2c}{9\kappa_{(1)}}\right)^{\frac{1}{8}} - \frac{7}{4}\frac{\kappa_{(2)}}{\kappa_{(1)}}\left(\frac{2c}{9\kappa_{(1)}}\right)^{-\frac{1}{8}} + o\left(\kappa_{(2)}^2\right) \;. \tag{5.17}$$

(v) For arbitrary dimension d

One can solve, in principle, polynomial equations of any degree in terms of *theta functions* of suitably high genus [9]. This applies both to (5.1) and (5.3). Hence, in principle, exact solutions can be constructed though it would be a very complicated task in practice.

One may note certain qualitative features easy to observe. Thus for example, the qualitative features described by (5.15) and (5.17) hold more generally. For the conditions concerning $\kappa_{(p)}$, stated after (II), the single real positive r_H tends to shrink due to $\kappa_{(2)}$ and $\kappa_{(p)}$ with $p > 2$, the black hole becoming smaller in radius.

In the preceding examples we have retained only $\kappa_{(1)}$ and $\kappa_{(2)}$ to illustrate basic features. For $d \geq 7$ one can include $\kappa_{(3)}$, for $d \geq 9$, $\kappa_{(4)}$, and so on. The equations become more difficult to solve but the general features appear already in our examples above.

In the illustrative examples considered in this Section, we were mostly concerned with equation (5.3) to find the horizon r_H. Concerning the evaluation of the function $L(r)$ on the other hand, one may note that in the equation (5.1) for L, up to $d = 10$ one has a quartic or an equation of lower degree for L. For $d = 11$ one has, retaining all possible nonzero contributions, for the first time, a quintic for L.

VI. MAXIMAL EXTENSIONS AND PERIODICITY FOR EUCLIDEAN SIGNATURE

We start by deriving a crucial ingredient in this context, determining both the maximal extension and the near–horizon geometry. For details we refer to Secs. **3** and **5** of [7] and references cited therein.

Set

$$r \;=\; r_H + \rho \,, \quad \rho \ll 1 \,. \tag{6.1}$$

Then, near the horizon, we define δ through

$$N(\rho) \;=\; 1 - L(\rho) \;=\; 2\delta\rho + o(\rho^2) \,. \tag{6.2}$$

It can be shown that

$$2\delta = \left(\frac{d-3}{r_H}\right)\left[\frac{\kappa_{(1)} + \frac{1}{4}(d-4)(d-5)\kappa_{(2)}r_H^{-2} + ... + \frac{1}{2^{n-1}n!}(d-4)...(d-2n-1)\kappa_{(n)}r_H^{-2(n-1)}}{\kappa_{(1)} + \frac{1}{2}(d-3)(d-4)\kappa_{(2)}r_H^{-2} + ... + \frac{1}{2^{n-1}(n-1)!}(d-3)...(d-2n)\kappa_{(n)}r_H^{-2(n-1)}}\right] \tag{6.3}$$

with r_H satisfying (5.3), where we have assumed that the first n $\kappa_{(p)}$, $p = 1,2,...,n$ are nonzero. The next step would be to substitute for r_H an explicit solution such as (5.7), (5.10) and so on. But the general expression (6.3) is particularly suitable for our present purpose.

If $\kappa_{(n)} = 0$ for $n > 1$, one obtains

$$2\delta \;=\; \left(\frac{d-3}{r_H}\right) \;=\; (d-3)\left(\frac{\kappa_{(1)}}{2c}\right)^{\frac{1}{d-3}} \,, \tag{6.4}$$

and usually, for $d = 4$, $\left(\frac{\kappa_{(1)}}{2c}\right)$ is defined as $(2M)^{-1}$.

Now we proceed to construct the Kruskal type maximal extension and the periodicity of the Euclidean metric (3.1) with Euclidean signature, namely

$$ds^2 = N dt^2 + N^{-1}dr^2 + r^2 d\Omega_{(d-2)}^2 \,. \tag{6.5}$$

We follow the standard procedure, which was generalised to one (single) member p of the hierarchy in Section **3** of [7]. Using (6.1) and (6.2), we set

$$r^\star = \int \frac{dr}{N} = \frac{1}{2\delta} \ln \rho + h(\rho) , \qquad (6.6)$$

in which the function $h(\rho)$ is not relevant for the singularity at the horizon.

We also introduce the coordinates (η, ζ) by

$$e^{2\delta r^\star} = \frac{1}{4}(\eta^2 + \zeta^2) , \qquad e^{i\delta t} = \left(\frac{\eta - i\zeta}{\eta + \zeta} \right)^{\frac{1}{2}} \qquad (6.7)$$

δ being given by (6.3). Here (η, ζ) provide the generalisation of Kruskal coordinates.

One obtains from (6.5) and (6.7)

$$ds^2 = (4\delta^2 e^{2\delta r^\star})^{-1} N(\eta, \zeta)(d\zeta^2 + d\eta^2) + r^2(\eta, \zeta) d\Omega^2_{(d-2)} , \qquad (6.8)$$

the factor of r^2 in the last term being implicitly a function of (η, ζ). The factor $(e^{2\delta r^\star})^{-1} N$ tends to unity as $\rho \to 0$, assuring maximal extension, there being neither a divergence nor a zero at the horizon.

One obtains from the second member of (6.7), for the period of t,

$$P = \frac{2\pi}{|\delta|} \qquad (6.9)$$

where δ is given by (6.3). For $n = 2$, i.e. only $\kappa_{(1)}$ and $\kappa_{(2)}$ nonzero, one obtains

$$P = \frac{4\pi r_H}{d - 3} \left[1 + \frac{(d - 1)(d - 4)\kappa_{(2)}}{4\pi\kappa_{(1)}r_H^2 + (d - 4)(d - 5)\kappa_{(2)}} \right] . \qquad (6.10)$$

The period P is inversely proportional to the Hawking temperature of the black hole. Substituting for r_H in (6.10) one obtains the full modification due to $\kappa_{(2)}$.

VII. COSMOLOGICAL CONSTANT AND POINT CHARGE

So far we restricted our attention to vacuum metrics. Let us now consider the more general case, keeping (4.8) and (4.9) in mind, namely (4.4)

$$V(r) = \frac{c}{r^{d-1}} - \frac{b}{r^{2(d-2)}} + \lambda .$$

We illustrate some basic features by setting

$$\kappa_{(3)} = \kappa_{(4)} = ... = \kappa_{(P)} = 0$$

in (4.3), whence (for $d > 4$)

$$\frac{\kappa_{(1)}}{2}(d - 2)\left(\frac{L}{r^2} \right) + \frac{\kappa_{(2)}}{8}(d - 2)(d - 3)(d - 4)\left(\frac{L}{r^2} \right)^2 = \frac{c}{r^{d-1}} - \frac{b}{r^{2(d-2)}} + \lambda . \qquad (7.1)$$

Setting

$$L = \hat{L} + \eta r^2 \qquad (7.2)$$

the left hand side of (7.1) becomes

10

$$\beta_1 \left(\frac{\hat{L}}{r^2} \right) + \beta_2 \left(\frac{\hat{L}}{r^2} \right)^2 + \beta_3 \tag{7.3}$$

with

$$\beta_1 = \frac{1}{2}(d-2)\kappa_{(1)} + (d-2)(d-3)(d-4)\eta\kappa_{(2)} \tag{7.4}$$

$$\beta_2 = \frac{1}{8}(d-2)(d-3)(d-4)\kappa_{(2)} \tag{7.5}$$

$$\beta_3 = \frac{1}{2}(d-2)\eta\kappa_{(1)} + \frac{1}{8}(d-2)(d-3)(d-4)\eta^2\kappa_{(2)} . \tag{7.6}$$

Setting

$$\beta_3 = \lambda \tag{7.7}$$

determines η. Then one solves

$$\beta_2 \left(\frac{\hat{L}}{r^2} \right)^2 + \beta_1 \left(\frac{\hat{L}}{r^2} \right) - \left(\frac{c}{r^{d-1}} - \frac{b}{r^{2(d-2)}} \right) = 0 . \tag{7.8}$$

One obtains, with sign conventions consistent with (5.4),

$$\hat{L}(r) = \left(\frac{\beta_1}{2\beta_2} \right) \left[\left(1 + \frac{4\beta_2}{\beta_1^2} \left(\frac{c}{r^{d-1}} - \frac{b}{r^{2(d-2)}} \right) \right)^{\frac{1}{2}} - 1 \right] r^2 , \tag{7.9}$$

so that $L(r)$ is given by (7.2).

For $b = 0$ and $\lambda \neq 0$, (7.9) for $\hat{L}$ has the same structure as for L in (5.4), with the constant coefficients now depending on λ, as given by (7.4)–(7.7). *It is easy to see that this feature will hold quite generally for all $\kappa_{(p)}$ (up to $p = P$).* Even in the presence of a point charge (with $b \neq 0$) the effect of λ can always be taken into account in this way.

When b is nonzero, it is seen from (4.8) to be positive for a real charge q. Hence for sufficiently small r, it can be seen from (7.9) that $\hat{L}$ (and hence L) becomes *complex*.

In a previous study [8] of the gravitational systems characterised by $\lambda = 0 = \eta$, but $b \neq 0$, and with a *single* nonvanishing $\kappa_{(p)}$, it was shown that such a point of transition from a real to a complex metric was situated *inside* the horizon(s) r_H of the generalised p–Reissner-Nordstrom type solutions. Hence in the *exterior* region the metric was always real. Also in [8], the possibility of a compatible real metric for the *interior* region was indicated. In the present work we will not undertake a parallel detailed study of (7.9).

VIII. DISCUSSION

We have studied black hole type solutions to generalised gravity in d-dimensions ($d \geq 5$). These systems consist of superpositions of successive higher order Gauss–Bonnet terms labeled by p ($2p \leq d$) which occur, amongst other gravitational terms, in the superstring inspired [1,29] gravitational system. In the present work, we have omitted the effects of all other fields, e.g. the dilaton, which also occur in superstring inspired systems [1,29]. So far, apart from introducing a cosmological constant and the Maxwell system (i.e. a point charge) in section **VII**, we have studied only the gravitational field with higher order terms. The scope can be broadened by including other fields relevant to string theory, and, by pursuing certain applications.

In the context of string theory, the fields to be added on to the gravitational systems are the dilaton, the Yang–Mills and the (Abelian) antisymmetric tensor fields.

Exact solutions including dilatons in this context were studied in [11] and were [12], but without higher order gravitational terms. The application of the efficient formalism given in the present work would enable the extension of these results [11,12] to the case of gravitational systems including higher GB terms.

Concerning the interaction of (the usual) Yang–Mills fields interacting with Einstein–Hilbert gravity in 4-dimensions, this has been intensely studied recently, and extensive references to it can be found in the review [13]. The extension of these considerations (with and without dilaton), involving generalised Yang–Mills systems interacting with generalised Einstein–Hilberts fields in higher (than 4)-dimensions, would be a very natural and efficient use our present results. Indeed, we have already considered generalised Yang–Mills fields on *fixed* generalised gravitational *backgrounds* in higher dimensions [14].

As to the introduction of antisymmetric tensor fields (in addition to the dilaton) to the usual gravity in the context of the low energy effective action of supergravity, this system supports solitonic solutions of supergravity [15]. As another application of our results, it would be very interesting to find out what the effect of adding higher-order gravitational terms would have on these solutions.

Finally, we mention some further possible applications of our results in the wake of earlier work in the literature where gravitational Lagrangians with higher terms were studied. These include applications to the elimination of ghosts [29], to the vanishing of the cosmological constant as a stable fixed point [16,17], and to the construction of gravitational instantons [3,4], as well as to cosmological solutions [18,19]. More recent work involving the first GB term [20], pertain to cosmological solutions [21] and to theories with noncompact extra diemsions [22–25] of gravity. The extension of these results [21–24] to include several higher-order GB terms, with and without the inclusion of the dilaton field, would constitute natural applications of our results.

Acknowledgements

Note added: After completion of this manuscript, many previous sources have been brought to our notice. To take account of these, we add the following supplementary references and explanations.

Most helpful is the extensive list of references in the review of Myers [26]. Without trying to be complete, we mention the following pioneering sources quoted there [27–29]. String generated cubic terms are studied in [30]. Maximal extensions have been studied systematically in [31]. Our results pertaining to maximal extensions are specifically concerned with the consequences of the supplementary $(d-2)$ dimensions and the higher curvature GB terms implemented through (6.3) in the standard Kruskal prescription in the $r-t$ plane.

Particularly relevant for us is [32]. The metric function (5.4) and the period (6.10) obtained by us as particular cases by setting $\kappa_{(p)} = 0$ for $p > 2$, match with eqns. (3) and (9) of [32], respectively for

$$\hat{\lambda} = \frac{\kappa_{(2)}}{2\kappa_{(1)}}(d-3)(d-4) \ . \tag{8.1}$$

For comparison with the *total Euclidean action*, eqn. (12) in [32], we briefly present the corresponding general result for our case

Total Euclidean Action: = Period × Area of unit sphere in $(d-2)$ dimensions × Radial Integral.

Period = $\frac{2\pi}{|\delta|}$; Area of unit sphere in $(d-2)$ dimensions, $A_{(d-2)} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$;

$$\text{Radial Integral} = \sum_p \frac{\kappa_{(p)}}{2p} \int_{r_H}^{\infty} R_{(p)} \ dr$$

$$= \sum_{p=1}^{P} \kappa_{(p)} \frac{(d-2)(d-3)...(d-2p+1)}{2^p p!} \int_{r_H}^{\infty} I_{(p)} \ dr \tag{8.2}$$

where

$$\int_{r_H}^{\infty} I_{(p)} dr = \int_{r_H}^{\infty} dr \; r^{d-2} \left(r \frac{d}{dr} + 2 \right) \left(r \frac{d}{dr} + 1 \right) \left(r^{d-2} \left(\frac{L}{r^2} \right)^p \right)$$
$$= \left[\frac{d}{dr} \left(r^d \left(\frac{L}{r^2} \right)^p \right) \right]_{r_H}^{\infty}$$
$$= \frac{2c}{(d-2)\kappa_{(1)}} \delta_{p,1} + \left((2\delta)p - \frac{(d-2p)}{r_H} \right) r_H^{d-2p} .$$

(8.3)

with

$$I_{(1)} = \frac{2c}{(d-2)\kappa_{(1)}} + 2\delta r_H^{d-2} - (d-2) r_H^{d-3} \tag{8.4}$$

$$I_{(2)} = 4 \; \delta \; r_H^{d-4} - (d-4) r_H^{d-5} \tag{8.5}$$

and so forth.

Note that the term independent of r_H in (8.4) comes from

$$\lim_{r \to \infty} \frac{d}{dr} \left(r^d \left(\frac{L}{r^2} \right) \right) = \lim_{r \to \infty} \frac{d}{dr} \left(\frac{2c}{(d-2)\kappa_{(1)} r^{d-1}} \cdot r^d \right) = \frac{2c}{(d-2)\kappa_{(1)}} .$$

[1] D.J. Gross and E. Witten, Superstring modification of Einstein equations, Nucl. Phys. **B 277**. 1 (1986)

[2] B. Zwiebach, Curvature squared terms and string theories, Phys. Lett. **B 156**, 915 (1985)

[3] A. Chakrabarti and D.H. Tchrakian, A compactification of, and instantons in, 4p-dimensional gravity, Phys. Lett. **B 168**, 187 (1986)

[4] G.M. O'Brien and D.H. Tchrakian, Spin-connection generalised Yang-Mills fields on double-dual generalised Einstein-Cartan backgrounds, J. Math. Phys. **29**, 1212 (1988)

[5] J.T. Wheeler, Symmetric solutions to the Gauss-Bonnet extended Einstein equations, Nucl. Phys. **B 268**, 737 (1986)

[6] J.T. Wheeler, Symmetric solutions to the maximally Gauss-Bonnet extended Einstein equations, Nucl. Phys. **B 273**, 733 (1986)

[7] A. Chakrabarti and D.H. Tchrakian, Antisymmetrised 2p-forms generalising curvature 2-forms and a corresponding p-hierarchy of Schwarzschild type metrics in dimensions $d > 2p + 1$, Adv. Theor. Math. Phys. **4**, 791 (2000); hep-th/9908128

[8] A. Chakrabarti and D.H. Tchrakian, Antisymmetrised 2p-forms generalising curvature 2-forms II: a p-hierarchy of Reissner-Nordstrom type metrics in dimensions d>2p+1; hep-th/0010011

[9] see for example, R. Bruce King, *Beyond the quartic equation*, Birkhäuser, 1996.

[10] R.C. Myers and M.J. Perry, Black holes in higher dimensional spce-times, Ann. Phys. **172**, 304 (1986)

[11] G. Gibbons and K. Maeda, Black holes and membranes in higher dimensional theories with dilaton fields, Nucl. Phys. **298**, 741 (1988)

[12] D. Garfinkle, G.T. Horowitz and A. Strominger, Charged black holes in string theory, Phys. Rev. **D 43**, 3140 (1991)

[13] M.S. Volkov and D.V. Gal'tsov, Gravitating non Abelian solitons and black holes with Yang–Mills fields, Phys. Rep. **319**, 1-83 (1999)

[14] Y. Brihaye, A. Chakrabarti and D.H. Tchrakian, finite energy/action solutions of p_1 Yang–Mills equations on p_2 Schwarzschild and de Sitter backgrounds in dimensions $d \geq 4$, J. Math. Phys. **41**, 5490 (2000)

[15] see for example Th. Mohaupt, Black holes in supergravity and string theory, hep-th/0004098.

[16] J. Madore, Kaluza–Klein theory with the Lanczos Lagrangian, Phys. Lett. **A 110**, 289 (1985)

[17] N. Deruelle and J. Madore, On the vanishing of the cosmological constant, Phys. Lett. **A 114**, 185 (1985)

[18] B. Giorgini and R. Kerner, Cosmology in 10 dimensions with the generalized graviational Lagrangian, Classical and Quantum Gravity, **5** 339 (1988)

[19] Müller-Hoissen, Spontaneous compactification with quadratic and cubic curvature terms, Phys. Lett **B 163** 106 (1985)

[20] P. Kanti, J. Rizos and K. Tamvakis, Singularity-free cosmological solutions in quadratic gravity, Phys. Rev. **D 59**, 083512 (1999)

[21] see S. Alexeyev, A. Toporensky and V. Ustiansky, The Nature of Singularity in Bianchi I Cosmological String Gravity Model with Second Order Curvature Corrections, gr-qc/0009020, and references therein.

[22] J.E. Kim, B. Kyae and H.M. Lee, Effective Gauss–Bonnet interaction in Randall–Sundrum compactification, hep-ph/9912344.

[23] I. Low and A. Zee, Naked singularity and Gauss–Bonnet term in brane world scenarios, hep-th/0004124.

[24] N.E. Mavromatos and and J. Rizos, String-inspired higher-curvature terms and the Randall–Sundrum scenario, hep-th/0008074.

[25] K.A. Meissner and M. Olechowski, Domain walls without cosmological constant in higher order gravity, hep-th/0009122

[26] R.C. Myers, Black holes in higher curvature gravity, in *Black Holes, Garvitational Radiation and the Universe: Essays in honour of C.V. Vishveshwara*, eds...., gr-qc/9811042

[27] D. Lovelock, The Einstein tensor and its generalisations, J. Math. Phys. **12** (1971);*ibid.* The four dimensionality of space and the Einstein tensor, **13**, 874 (1972)

[28] D.G. Boulware and S. Deser, String generated gravity models, Phys. Rev. Lett. **55**, 2656 (1985)

[29] B. Zumino, Gauge theories in more than four dimensions, Phys. Rep. **137**, 2656 (1985)

[30] M.C. Bento and O. Bertolami, String generated gravity models with cubic curvature terms, Phys. Lett. **B 228**, 348 (1989)

[31] T. Klösch and T. Srtobl, Classical and quantum gravity in 1+1 dimensions, Class. Quan. Grav. **13**, 965 (1996) (corrigendum in *ibid.* **14**, 825 (1997)); *ibid.* **14**, 1689 (1997)

[32] R.C. Myers and J.Z. Simon, Black hole thermodynamics in Lovelock gravity, Phys. Rev. **D 38**, 2434 (1988)